Self-averaging of an order parameter in randomly coupled limit-cycle oscillators

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In our recent paper [Phys. Rev. E **58**, 1789 (1998)] we found notable deviations from a power-law decay for the "magnetization" of a system of coupled phase oscillators with random interactions claimed by Daido in Phys. Rev. Lett. **68**, 1072 (1992). For another long-time property, the Lyaponov exponent, we found that his numerical procedure showed strong time discretization effects and we suspected a similar effect for the algebraic decay. In the Comment to our paper [preceding paper, Phys. Rev. E **61**, 2145 (2000)] Daido made clear that the power law behavior was only claimed for the sample averaged magnetization $[Z]$ and he presented new, more accurate numerical results which provide evidence for a power-law decay of this quantity. Our results, however, were obtained for Z itself and not for $[Z]$. In addition, we have taken the intrinsic oscillator frequencies as Gaussian random variables, while Daido in his new and apparently also in his earlier simulations used a deterministic approximation to the Gaussian distribution. Due to the differences in the observed quantity and the model assumptions our and Daido's results may be compatible.

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The investigation of interacting biological oscillators was introduced by Winfree [1] several decades ago. Building on Winfree's idea of a phase description, Kuramoto introduced a simple model of interacting phase oscillators [2–5], which could be solved analytically in the limit of $N\to\infty$ coupled oscillators. The dynamics of the phases of the oscillators are described by coupled first order differential equations:

$$\dot{\phi}_j(t)=\omega_j+\sum_i J_{ij}\sin[\phi_i(t)-\phi_j(t)], \qquad (1)$$

where $\phi_j(t)$ denotes the phase of the jth oscillator, $\dot{\phi}_j(t)$ its time derivative, ω_j its natural (undisturbed) frequency, and J_{ij} the strength of the interactions between oscillator i and j, which in this model is assumed to be an uniform ferromagnetic all-to-all interaction, i.e., $J_{ij}=K/N$, with interaction strength $K\geqslant 0$. For $N\to\infty$, and a distribution function of frequencies $f(\omega)=\Sigma_{j=1}^{N}\delta(\omega-\omega_j)/N$, which is assumed to be symmetric, Kuramoto derived an analytic expression for the order parameter $Z=\Sigma_{j=1}^{N}\exp(i\phi_j)/N$. This order parameter is identical to the magnetization in the XY model of two-dimensional spins [6] and describes the synchronization of the oscillators.

Daido [7] investigated a version of (1) with spin-glass-type interactions analogous to SK models of spin glasses [9, 8]. J_{ij} were chosen as symmetric Gaussian random variables with zero mean and standard deviation $J/\sqrt{N}$, i.e., they obey $[J_{ij}]=0$ and $[J_{ij}J_{kl}]=\delta_{i,k}\delta_{j,l}J^2/N$ with $[\cdot]$ denoting the 'quenched' average over the random variables, and $J\geqslant 0$ denoting the interaction strength. The frequencies ω_j are distributed according to a Gaussian distribution function $f(\omega)=\exp(-\omega^2/2)/\sqrt{2\pi}$.

A continuous distribution function $f(\omega)$ is only well-defined for $N\to\infty$ or for random variables ω_j. Although not stated in [7], in his numerical calculations (with finite N) Daido, in contrast to us [10], used nonrandom frequencies [see Eq. (3) in his comment [11]].

Daido investigated the decay of $Z(t)$ from the initial condition $Z(0)=1$ [i.e., $\phi_j(0)=0$ for all j] with numerical calculations, finding a transition from exponential to power-law decay. Since in his paper [7] Daido did not clearly distinguish between $|Z|$ and the sample averages $|[Z]|$ and $[|Z|]$, one could expect that by self-averaging an identical behavior is found for all three quantities [see also the remarks (11) and (12) in Daido's comment [11]].

In our paper [10] we actually investigated a generalization of the model in [7]. We analyzed Eq. (1) with Gaussian random interaction strengths J_{ij} and random frequencies ω_j,

$$[J_{ij}]=0,\ [\omega_i]=\omega_0,$$
$$[J_{ij}J_{kl}]=(\delta_{ik}\delta_{j,l}+\delta_{i,l}\delta_{j,k}\eta)J^2/N,\ \ [\omega_i\omega_j]=\mu^2\delta_{i,j}. \qquad (2)$$

For $\eta=1$ the interaction is symmetric, for $\eta=-1$ antisymmetric, and for $\eta=0$ the random variables J_{ij} and J_{ji} are uncorrelated. From the motivation of this model by interacting biological oscillators (see also the first paragraph of [11]) random frequencies ω_j are certainly a natural choice.

We were mainly concerned with deriving a dynamic mean field theory and investigating the dynamic properties such as the occurence of chaos. Among other quantities we also investigated the decay of the order parameter Z (denoted m in [10]). Since the quenched average $[Z]$ may not be relevant

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for a system of coupled oscillators (one realization of the random variables), unless Z is self-averaging, we investigated Z itself. For symmetric interactions ($\eta=1$) we did find deviations from a power law for all but one critical interaction strength.

We showed that the numerical procedure used by Daido (Euler integration scheme with time discretization $\Delta t=10^{-2}$) caused strong time discretization effects for another long-time property, the Lyaponov exponent, which were not present for the numerical procedure of higher-order and smaller time discretization used in our calculations, and we suspected a similar effect for the algebraic decay.

In his comment Daido presents new results with a numerical procedure of higher order, a larger number of oscillators and more realizations of the random variables, which provide evidence that $|[Z(t)]|$ indeed obeys a power law.

Since it is not claimed by Daido that $Z(t)$ is also obeying a power law, and also slightly different systems were investigated (random vs nonrandom natural frequencies ω_j) there are no contradictions with our results. As the results in our paper [10] and Daido's comment [11] suggest that Z might not be self-averaging, future research should be devoted to this question. Also the question whether the (non-) randomness of the natural frequencies ω_j leads to different results deserves further investigations. Finally we emphasize that we agree with Daido on the importance of finite size effects in this system (see, e.g., our Fig. 3 in [10]). In our opinion, however, the differences between Daido's results and ours are not a question of finite-size effects but should rather be considered as a hint to a more complex, non-self-averaging behavior in this system.

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